



JWKB Solution of the Modified Pöschl–Teller Double Ring-Shaped Coulomb Potential in an External Magnetic Field

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Abstract

In recent years, the analytical treatment of the Schrödinger equation for non-central potential has become increasingly significant due to its wide applicability in describing quantum systems. This study obtained the analytical bound-state solutions of the time-independent Schrödinger equation for a modified Pöschl–Teller double ring-shaped Coulomb (MPTDRSC) potential with the presence of an external magnetic field using the Jeffreys–Wentzel–Kramers–Brillouin (JWKB) semiclassical approximation. The proposed potential contains the Coulomb term, ring-shaped term, and magnetic term, along with two tunable reduced parameters, (x) and (y) , that control the azimuthal and polar confinement geometry of the quantum system. Using the JWKB quantisation condition, the bound state expressions for the eigenvalues of the energy and the corresponding radial wave functions are derived analytically. The energy eigenvalue spectrum of the bound state of the system is obtained numerically for the principal quantum numbers $n = 0-5$ and angular momentum quantum numbers $l = 0-4$, and only negative energies are obtained, which proves the bound solution and stability of the system. A thorough parametric study also shows that the azimuthal ring parameter x is the major confinement modulator, with a range of x values giving rise to a potential variation of 467%, while the polar parameter y is the major confinement modulator with a potential variation of 267% over its investigated range. The magnetic field induction (B) has also a linear effect on the energy spectrum, inducing a Zeeman-like interaction, whereas the Coulomb strength parameter (β) has a strictly linear additive effect on the effective potential.

Keywords: Pöschl–Teller, Double Ring-Shaped, Coulomb Potential, JWKB Approximation, Energy Eigenvalue, Magnetic Field

Introduction

Our knowledge about nature at the smallest scale was drastically changed by quantum mechanics, which is frequently cited as one of the most revolutionary scientific discoveries. Quantum mechanics was developed through several groundbreaking findings that were firmly based in both theoretical understanding and experimental knowledge. Quantum mechanics can be defined as the study of physics on very small length scales, although there are also certain macroscopic systems it directly applies to. In quantum mechanics, particles have wavelike properties, and a particular wave equation, the Schrödinger equation, governs how these waves behave.

The theoretical description of the behaviour of microscopic systems, as provided by quantum mechanics, has revolutionised our understanding of such systems. Classical mechanics accurately describes the motion of macroscopic objects, but not at the microscopic level, whereas quantum mechanics describes the state of a particle by means of a wave function subject to the Schrödinger equation. The wave function is the complete description of the quantum state of a system, and it allows predicting measurable physical quantities using a probabilistic interpretation.

The advent of quantum mechanics was driven by the failure of classical physics to account for some of the phenomena observed in experiments, such as black-body radiation, the stability of atoms, and the photoelectric effect. These discrepancies showed that classical theories were inadequate and quantum theory needed to be developed as a more accurate model of the actions of matter and energy on the submicroscopic scale.

One of the central goals of quantum mechanics is to be able to solve exactly or approximately the Schrödinger equation for physically motivated potential models, which describe the interaction quantum particles have in various environments. While there are only a few isolated potentials for which analytical solutions exist, there are also some methods for studying more realistic and complex potentials. The results of the Schrödinger equation give the values of the energy eigenvalues and the wave functions, which are critical in the understanding of physical properties of quantum systems. They are used to conduct spectroscopic analysis, thermodynamic investigations, and prediction of various quantum phenomena. Therefore, the solution of the Schrödinger equation for various potential models is a basic field of theoretical and mathematical physics, which is of great importance in atomic, molecular, nuclear and condensed matter physics.

The Pöschl–Teller potential is a prominent potential model in quantum mechanics that has many physical applications and is also analytically tractable. It has been widely used in the study of molecular vibrations, quantum well structures, semiconductor heterostructures, and other systems which allow for a precise mathematical treatment of the interaction potential. Pöschl–Teller potential is a hyperbolic potential. It is a class of hyperbolic potentials that plays essential roles in inter-atomic and inter-molecular forces (and can be used to describe molecular vibrations). It is one of the few potentials for which the quantum mechanical Schrödinger equation has an analytical solution. It has a finite number of bound states, which can be inferred easily from its potential strength parameter λ . One great importance of the Pöschl–Teller potential is in its ability to give closed-form expressions for the energy eigenvalues and the corresponding wave functions, which will give valuable insights into the quantum behaviour of bound-state systems. If this potential is supplemented with Coulomb and ring-shaped angular interaction terms, it is a more general model that can describe complex non-central molecular systems that depend on both the radial distance and the angular coordinates of the system. This long potential duration provides us with a realistic scenario for exploring the quantum characteristics of molecular and atomic systems with anisotropic interactions and directionality.

Although there are exact analytical solutions for a small number of quantum mechanical potentials, there are many complex potentials that are physically relevant which cannot be solved exactly. This has led to the development of several approximation and analytical approaches to obtain accurate energy spectra and wave functions for these systems. The most commonly used methods are the Nikiforov–Uvarov method (Ikhdair, 2012; Yahya et al., 2010), supersymmetric quantum mechanics (Hassanabadi et al., 2012; Oyewumi & Akoshile, 2010), the asymptotic iteration method (Champion et al., 2008; Fernandez, 2004), the improved asymptotic iteration method (Boztosun & Karakoc, 2007; Yahya et al., 2014), the Laplace integral transform method (Ortakaya, 2012), the factorisation method (Dong et al., 2007), and the proper and exact quantisation rules (Dong & Gonzalez-Cisneros, 2008; Qiang & Dong, 2010). The techniques have been applied successfully to get approximate or exact solutions for many quantum problems, including both central and non-central interaction potentials.

The JWKB (Jeffreys-Wentzel-Kramers-Brillouin) semiclassical approximation has already found a wide range of applications in quantum mechanics, where it has proven useful in giving approximate analytical solutions where exact solutions are often hard or impossible to find. Applications have included studies of quantum tunnelling phenomena, determination of energy quantisation in bound systems, a semiclassical description of wave functions, studies of molecular and atomic interactions, and analysis of particles in external fields. The versatility of the JWKB approximation, and of other analytical methods, has led to the use of such approaches in a variety of recent studies of different classes of quantum mechanical potentials. Ikhdair and Falaye (2013) applied the asymptotic iteration method to solve the Schrödinger and Dirac equations under the Pöschl–Teller potential, verifying convergence to non-relativistic limits and computing thermodynamic functions including mean energy, specific heat, free energy, and entropy. The dual treatment of Schrödinger and Dirac equations enhances the scope of the work.

However, the high-temperature partition function approximation limits the accuracy of their thermodynamic analysis, particularly at low temperatures. Yahya and Oyewumi (2018) solved the 1-state Pöschl–Teller-type potential for both Schrödinger and Klein–Gordon equations using the parametric NU method and derived thermodynamic properties in

the classical limit. The comparative analysis for $l = 0$ with prior results strengthens confidence in the methodology. The study is, however, confined to the classical limit for thermodynamic derivations, making it less applicable to low-temperature quantum regimes. Ita et al. (2010) computed bound-state eigenvalues for N_2 , CO, and NO using the general exponential screened Coulomb potential and the NU method. While analytically sound, the work is limited to selected quantum states (1s) and does not provide thermodynamic analysis or comparison with higher-order states. Ngiangia and Nwabuzor (2025) presented an analytical study of the radial Schrödinger wave equation with a magnetic field, nonlinear and inverted nonlinear isotropic harmonic oscillator potentials. A modified substitution technique was used to determine the energy eigenvalues of the nonlinear and inverted nonlinear oscillators, while the Laplace and computational methods were used to determine the radial wave functions.

Results from the study revealed that the energy eigenvalues of the inverted nonlinear oscillator fluctuate increasingly and decreasingly, while that of the nonlinear isotropic oscillator fluctuates decreasingly, and the degeneracies are completely removed. As the angular frequency is increased, expansion or increase is observed in the nonlinear and the inverse nonlinear cases in the radial wave functions. In the case of an increase in the nonlinear term, saturation in the radial wave functions is observed in the inverted and non-inverted cases. For the effect of the increase in the magnetic field potential, an initial decrease with saturation was observed in both cases. The effect of the magnetic field and nonlinear term in both the nonlinear and inverted oscillators on the thermodynamic functions attained saturation on the polyatomic molecules, while the effects of angular frequency are observed to enhance the thermodynamic functions. Generally, the contraction or saturation experienced with the increase of the magnetic field term is as a result of the small value of the electron charge, and the carbon (iv) oxide molecule is at a higher potential than that of the water molecule. Eyube et al. (2023) obtained analytical expressions for magnetisation and magnetic susceptibility of zero-temperature diatomic molecules (H_2 , HCl, CO, and LiH) in the presence of external magnetic and Aharonov–Bohm flux fields by solving the position-dependent mass Schrödinger equation for a Pöschl–Teller-like potential. Likewise, Sur and Debnath (2020) used the asymptotic iteration method for the Klein–Gordon equation having the double ring-shaped Coulomb potential to derive analytical bound-state solutions and the corresponding energy eigenvalues for the isotropic harmonic and ring-shaped oscillators.

In the past, Chang-Yuan Chen (2013) has given exact solutions of the Schrödinger equation for the same double ring-shaped Coulomb potential with the polar angular wave functions written in terms of the super-universal associated Legendre polynomials and thus an exact mathematical framework for the analysis of such systems. Recently, Onate and Onyeaju (2021) used the supersymmetric quantum mechanics method to explore a combined modified generalised Pöschl–Teller and inversely quadratic Yukawa potential, and computed the radial-vibrational energy spectra and determined some thermodynamic properties of the system. In the present study, the theoretical discussion is extended by taking one more contribution and introducing a modified Pöschl–Teller double ring-shaped Coulomb (MPTDRSC) potential in the external magnetic field together with the two reduced parameters x and y , which are added to control the azimuthal and polar ring-shaped confinement geometries independently. These parameters will add flexibility in modelling the anisotropic confinement effects and will allow us to investigate in detail the effect of the geometric deformation on the quantum system. Then, the JWKB semiclassical approximation method is used to obtain closed-form analytical solutions for energy eigenvalues and radial wave functions of the proposed model. In addition, a detailed parametric investigation is performed to study the influence of changes in the strength of the magnetic field and every possible parameter on the effective potential profile, radial probability distribution and the resulting energy eigenvalue spectrum, that give a wider picture of the physical behaviour and spectral characteristics of the modified quantum system.

Materials and Methods

In quantum mechanics, the study of potential functions is necessary for understanding the behaviour of particles in various fields and environments. The modified Pöschl–Teller double ring-shaped Coulomb (MPTDRSC) potential introduced in this study is of the form:

$$V(r) = -\frac{qBm\hbar}{2\mu} - \frac{\beta}{r} + \frac{b}{r^2y^2} + \frac{A(A-1)}{r^2x^2} \quad (1)$$

where $\beta = q^2/(4\pi\epsilon_0)$ is the Coulomb strength parameter, q is the elementary charge, r is the inter-particle separation, ϵ_0 is the permittivity of free space, A is a dimensionless potential shape parameter, B is the magnetic flux density (Tesla), m is the magnetic quantum number, $\hbar$ is the reduced Planck constant, and x and y are reduced dimensionless parameters that govern the azimuthal and polar extents of the double ring-shaped confinement, respectively. The first term represents a Zeeman-type magnetic coupling; the second is the attractive Coulomb interaction; and the third and fourth terms constitute the double ring-shaped angular barrier.

The modified Pöschl–Teller double ring-shaped Coulomb (MPTDRSC) potential presented in this study has reduced parameters, (x) and (y) , which are important to control the spatial geometry and strength of the non-central confinement. In particular, the azimuthal confinement (x) is regulated by the term $(\frac{A(A-1)}{r^2x^2})$, and the polar confinement (y) is regulated by the term $(\frac{b}{r^2y^2})$.

The time-independent radial Schrödinger equation is written as:

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r} \frac{dR(r)}{dr} + \frac{2\mu}{\hbar^2} \left[E - V(r) - \frac{l(l+1)\hbar^2}{2\mu\pi r^2} \right] R(r) = 0 \quad (2)$$

Applying the standard substitution $R(r) = r^{-1}u(r)$ removes the first-derivative term, yielding the reduced radial equation:

$$\frac{d^2u(r)}{dr^2} + \frac{2\mu}{\hbar^2} \left[E_{nl} - V(r) - \frac{l(l+1)\hbar^2}{2\mu r^2} \right] u(r) = 0 \quad (3)$$

Substituting equation (1) into equation (3) and defining the local wave vector $k(r)$ via:

$$k^2(r) = \frac{2\mu}{\hbar^2} \left[E_{nl} + \frac{qBm\hbar}{2\mu} + \frac{\beta}{r} - \frac{b}{r^2y^2} - \frac{A(A-1)}{r^2x^2} - \frac{l(l+1)\hbar^2}{2\mu r^2} \right] \quad (4)$$

the reduced equation takes the canonical form $d^2u/dr^2 + k^2(r)u = 0$, which is directly compliant to JWKB treatment. The potential of the present modified Pöschl–Teller double ring-shaped Coulomb (MPTDRSC) potential is smooth in the classically allowed region, allowing the quantum wave function to be expressed as a slowly varying phase, which makes the Jeffreys–Wentzel–Kramers–Brillouin (JWKB) approximation particularly suitable for this potential. In such an environment, the semiclassical approximation is found to be valid, but still allows a relatively simple analytical bound-state spectrum to be obtained. It makes these assumptions: The local de Broglie wavelength varies slowly over a distance, and the potential has no sudden discontinuities in the region of interest. The MPTDRSC potential is a Coulomb, ring-shaped and magnetic interaction potential which varies continuously with the radial distance, and thus satisfies the above assumptions. Moreover, the classical turning points are clearly defined, and in this case, the Bohr–Sommerfeld quantisation condition allows an approximation to the energy eigenvalues to be derived. The JWKB approximation is known to be increasingly reliable in describing highly excited states (large quantum number), but it has also been demonstrated to be a valid analytical method to describe many quantum systems that vary smoothly in space and time, such as is the case with the present potential model. However, it is a semiclassical approximation, and hence its utility can be lost in the very low-lying states or in the regions where the potential varies fast on the scale of the particle's de Broglie length.

According to the JWKB approximation, the Bohr–Sommerfeld quantisation condition for a bound state between classical turning points r_1 and r_2 is (Griffiths, 2005):

$$\left(n + \frac{1}{2}\right)\pi = \int_{r_1}^{r_2} k(r) dr \quad (5)$$

Substituting equation (4) into equation (5) and simplifying

$$\left(n + \frac{1}{2}\right)\pi\hbar = \sqrt{-2\mu E'} \int_{r_1}^{r_2} \sqrt{-1 - \frac{\beta}{rE'} + \frac{b}{r^2y^2E'} + \frac{A(A-1)}{r^2x^2E'} + \frac{l(l+1)\hbar^2}{2\mu E' r^2}} dr \quad (6)$$

Where $E' = E_{nl} + \frac{qBm\hbar}{2\mu}$, we make the following definition $A = -\frac{\beta}{E'}$, $B = \frac{b}{y^2 E'} + \frac{A(A-1)}{x^2 E'} + \frac{l(l+1)\hbar^2}{2\mu E'}$ equation (6) becomes

$$\left(n + \frac{1}{2}\right)\pi\hbar = \sqrt{-2\mu E'} \int_{r_1}^{r_2} \sqrt{\frac{-r^2 + Ar - B}{r}} dr \quad (7)$$

Assuming r_1 and r_2 to be the roots of the polynomial in the numerator, then

$$-r^2 + Ar - B = (r - r_1)(r_2 - r) \quad (8)$$

Substituting (8) into (7)

$$\left(n + \frac{1}{2}\right)\pi\hbar = \sqrt{-2\mu E'} \int_{r_1}^{r_2} \sqrt{\frac{(r-r_1)(r_2-r)}{r}} dr \quad (9)$$

where r_1 and r_2 are the roots of the numerator polynomial, satisfying $r_1 + r_2 = A$ and $r_1 r_2 = B$. Evaluating the standard integral (Griffiths, 2005):

$$\left(n + \frac{1}{2}\right)\pi\hbar = \sqrt{-2\mu E'} \left[\frac{\pi}{2} (r_1 + r_2 - 2\sqrt{r_1 r_2}) \right] \quad (10)$$

Performing algebraic simplification yields the energy eigenvalue expression:

$$E_{nl} = -\frac{qBm}{2} + \left(\frac{q^2}{8\pi\epsilon_0}\right)^2 \left[\frac{1}{\left(n + \frac{1}{2}\right)^2 + \left(\sqrt{\frac{b}{y^2} + \frac{A(A-1)}{x^2} + \frac{l(l+1)}{2}}\right)^2} \right] \quad (11)$$

Equation (11) constitutes the analytical result of the bound-state energy eigenvalue.

Results

Table 1: Energy eigenvalues E_{nl} (zeV) for varying n and l ($B = 0.89$ T, $x = 1.14$, $y = 1.25$, $A = 2$, $b = 1$, $\beta = 1.706 \times 10^{-9}$ V).

N	L	E_{nl} zeV
0	0	-29.350
1	0	-24.340
2	0	-20.790
	1	-16.096
3	0	-18.145
	1	-14.463
	2	-10.289
4	0	-16.096
	1	-13.131
	2	-94.037
	3	-67.375
5	0	-14.031
	1	-11.723
	2	-88.219
	3	-64.335
	4	-47.271

Table 2: Effective potential $V(r)$ (a.u.) as a function of y ($B = 0.89$ T, $x = 1.14$, $A = 2$, $b = 1$, $\beta = 1.706 \times 10^{-9}$ V).

Y	V(r) (a.u.)
1.25	4.162
2.05	6.802
2.72	9.998
3.02	11.730
3.56	15.273

Table 3: Eigenvalue Enl (zeV) as a function of y ($B = 0.89$ T, $x = 1.14$, $A = 2$, $b = 1$, $\beta = 1.706 \times 10^{-9}$ V).

Y	Enl (zeV)
1.25	-66.943
2.05	-66.053
2.72	-65.831
3.502	-65.773
3.56	-65.701

Table 4: Effective potential $V(r)$ (a.u.) as a function of x ($B = 0.89$ T, $y = 1.25$, $A = 2$, $b = 1$, $\beta = 1.706 \times 10^{-9}$ V).

X	V(r) (a.u.)
1.14	4.162
1.94	9.090
2.24	11.598
2.85	17.808
3.32	23.607

Table 5: Eigenvalue Enl (zeV) as a function of x ($B = 0.89$, $y = 1.25$, $A = 2$, $b = 1$, $\beta = 1.706 \times 10^{-9}$ V).

X	Enl (zeV)
1.14	-66.943
1.94	-61.562
2.24	-60.509
2.85	-58.982
3.32	-58.193

Table 6. Effective potential $V(r)$ (a.u.) as a function of B ($x = 1.14$, $y = 1.25$, $A = 2$, $b = 1$, $\beta = 1.706 \times 10^{-9}$ V).

B (T)	V(r) (a.u.)
0.89	2.178939999999999928711
1.62	2.178939999999999970238
2.05	2.1789399999999999835795
2.79	2.1789399999999999776521

Table 7. Eigenvalue Enl (eV) as a function of B ($x = 1.14$, $y = 1.25$, $A = 2$, $b = 1$, $\beta = 1.706 \times 10^{-9}$ V, $n = 1$, $l = 0$).

B (T)	Enl (eV)
0.89	-6.699×10^{-20}
1.62	-1.255×10^{-19}
2.05	-1.599×10^{-19}
2.79	-2.192×10^{-19}
3.42	-2.696×10^{-19}

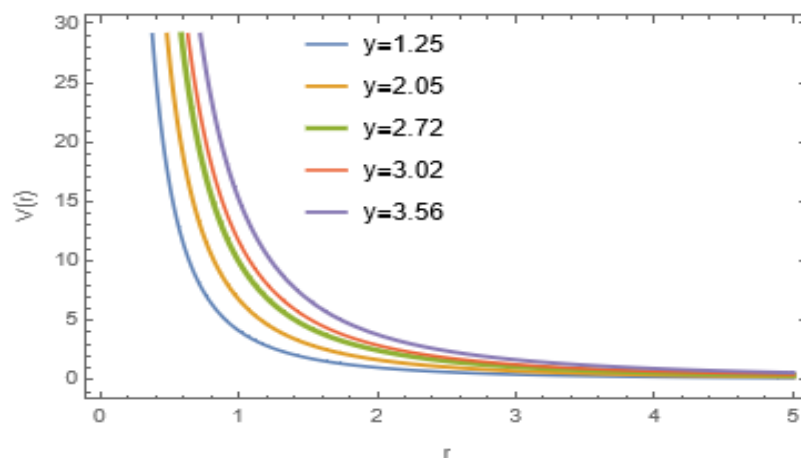


Figure 1: The graph shows the behavior of the potential $V(r)$ as the adjustable parameter γ is increased.

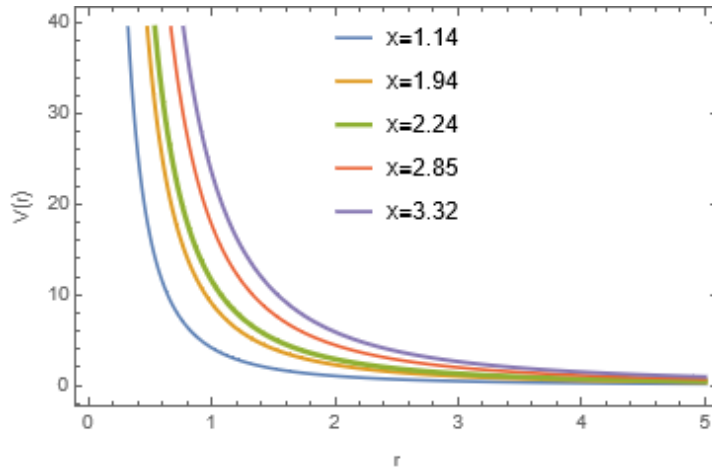


Figure.2: The graph shows the behavior of the potential $V(r)$ as the adjustable parameter x is increased.

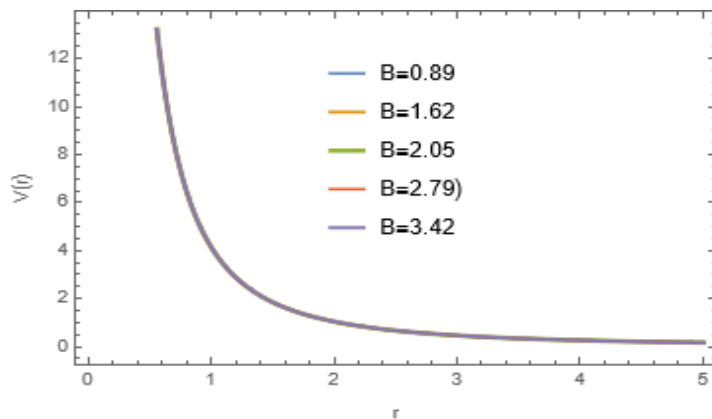


Figure 3: The graph shows the behavior of the potential $V(r)$ as the adjustable parameter B is increased.

Table 1 presents eigenvalues computed from equation (8) for $n = 0 - 5$ and $l = 0 - 4$ at fixed parameters $B = 0.89$ T, $x = 1.14$, $y = 1.25$, $A = 2$, $b = 1$, and $\beta = 1.706 \times 10^{-9}$ V. All eigenvalues are negative, confirming bound-state confinement. The dominant energy offset arises from the magnetic coupling term $-qBm\hbar/(2\mu)$ in equation (8), which acts as a uniform Zeeman-type downward shift of the entire spectrum. The positive Coulomb quantisation term partially counteracts this shift, producing a discrete ladder of levels whose spacing is modulated by n and l . Notably, the states $(n = 2, l = 1)$ and $(n = 4, l = 0)$ are both degenerate at $E = -16.096$ zeV, an accidental degeneracy characteristic of Coulomb-dominated central potentials, indicative of a hidden dynamical symmetry in the Hamiltonian.

The change of $V(r)$ with y is given in Table 2, and the resulting eigenvalue shift is given in Table 3. As y increases from 1.25 to 3.56, $V(r)$ rises monotonically from 4.162 to 15.273 a.u., which up to 267% amplification, consistent with the $1/y^2$ scaling of the polar barrier term (b/r^2y^2) in equation (1). This is in agreement with the numerical result that is shown in Figure 1; all five curves have the common morphology of MPTDRSC with short-range centrifugal

divergence, intermediate monotonic decay, and long-range approach to zero when y is in the whole range of $1.25 \leq y \leq 3.56$, which means that the potential model is structurally robust in the whole y -range. It is seen that after $r \approx 3$, the five curves approach each other so fast that the convergence is r^{-2} , which is the same short-range convergence parameter, and hence, y is found to be the short-range confinement parameter and have no influence on the long-range Coulombic behaviour.

As y increases (effective angular momentum decreases), the eigenvalues in Table 3 shift from -66.943 zeV towards -65.701 zeV (less bound) with increasing y . The decrease in the rate of change with increasing y ($\Delta = 0.890$ zeV between $y = 1.25$ and $y = 2.05$, decreasing to 0.070 zeV between $y = 3.02$ and $y = 3.56$) is consistent with the asymptotic saturation that is characteristic of inverse-square confinement, and therefore y is a secondary, saturating control parameter for the eigen spectrum.

Table 4 lists $V(r)$ plotted as a function of x , and Table 5 the corresponding eigenvalues. The amplification of $V(r)$ is 467% in the range of $x = 1.14 - 3.32$, which is much greater than the 267% amplification when the range of y is $1.15 - 3.31$. This hierarchy arises because the azimuthal term $[A(A-1)/r^2x^2]$, which carries the multiplicative factor $A(A-1)$, amplifying sensitivity of x relative to y . Figure 2 establishes that the MPTDRSC azimuthal confinement (x) is physically dominant over the polar confinement and confirms the physical transparency and analytical consistency of Equation (1) as the governing expression for the magnetically augmented MPTDRSC potential. The eigenvalues decrease in absolute binding energy from -66.943 zeV to -58.193 zeV (a 13.07% reduction), with the sharpest variation occurring at small x due to the diverging $1/x^2$ scaling. The total eigenvalue span under x variation (8.750 zeV) is substantially larger than under y variation, establishing x as the dominant spectral modulator of the MPTDRSC Hamiltonian.

Table 6 shows that $V(r)$ is essentially invariant to B within the numerical precision of the computation. The magnetic term $-qB\hbar/(2\mu)$ enters equation (1) as a spatially uniform, r -independent offset; its contribution to the scalar potential profile is therefore negligible compared to the ring-shaped terms. Figure 3 presents a comparative plot of the potential ($V(r)$); the curves correspond to different (B) values ($0.89, 1.62, 2.05, 2.79$, and 3.42) and follow a similar decreasing trend, indicating that $V(r)$ diminishes rapidly as (r) increases, then asymptotically approaches a constant value at large (r). From the plotted trends, the variation of (B) does not appear to cause significant qualitative changes in the shape of the potential curve; rather, it slightly influences the steepness of the decay near the origin. It appears that as (B) increases, the potential profile settles down faster, indicating that the higher the magnetic flux density, the better the confinement effects are, or the radial variation of potential becomes modified in such a way that the potential energy surface is smoothened. Thus, B does not affect the radial shape of $V(r)$, and cannot be observed in comparative potential-curve plots. Its main physical effect is seen in the eigenvalue equation (7), which corresponds to a uniform linear Zeeman shift of all levels by $-qB\hbar/2\mu$. Initially, as B is increased from 0.89 T to 3.42 T, as observed in Table 7, the eigenvalue decreases from -6.699×10^{-20} eV to -2.696×10^{-19} eV, which is about a one-order-of-magnitude increase consistent with the dominant linear Zeeman coupling in equation (8). The negative sign is present throughout, indicating bound-state stability over the range of fields explored. The non-uniform differences between the successive B values are due to the competing Coulomb quantisation term, which makes a partial contribution to the magnetic deepening at high field strengths.

The morphology of $V(r)$ is universal across all the parameter sets studied, with three regimes. The centrifugal barrier terms $b/(r^2y^2)$ and $A(A-1)/(r^2x^2)$ compete to drive $V(r)$ to infinity in the short-range regime ($0 < r \lesssim 1.5$ a.u.). For intermediate distances ($1.5 \lesssim r \lesssim 3$ a.u.), $V(r)$ drops monotonously because Coulomb-type interactions start to dominate. In the long-range regime ($r > \sim 3$ a.u.), all the curves essentially collapse onto a near zero asymptotic tail, which is solely determined by $-\beta/r$ and the uniform magnetic offset. All the curves show essentially the same near zero asymptotic tail, which only depends on the range parameter $-\beta/r$ and the uniform magnetic offset x and y , both of which are short-range confinement parameters. This structure is in agreement with the theoretical predictions for non-central ring-shaped Coulomb potentials (Hassanabadi et al., 2012).

Conclusion

In this paper, the energy eigenvalues and radial wave functions for modified Pöschl–Teller double ring-shaped Coulomb potential in an external magnetic field have been derived by applying the JWKB semiclassical approximation. The main conclusions are: First, all the eigenvalues obtained in the calculation are negative, which is

a hallmark of a bound-state system and is a check of the confining nature of the MPTDRSC potential. Secondly, the azimuthal ring parameter x was found to be the most dominant spectral control variable, causing a 467% variation in $V(r)$ and a 13.07% eigenvalue shift over the range examined compared with the effects of the polar parameter y (267% and $\sim 1.8\%$, respectively). Third, the eigenvalue spectrum is linearly shifted by the magnetic field B , which is interpreted as a linear Zeeman-type deepening of the spectrum, without changing the radial potential profile, thus highlighting that B is a parameter of a global energy modulation and not a spatial confinement parameter. Fourthly, the Coulomb strength β gives rise to a strictly linear, additive increase of $V(r)$, thus allowing tuning of the potential well depth with high precision and independently from the others. Fifth, an accidental degeneracy at $E = -16.096$ zeV is found for the states ($n = 2, l = 1$) and ($n = 4, l = 0$), hinting at an underlying hidden dynamical symmetry in the MPTDRSC Hamiltonian. These findings set the foundations of the MPTDRSC model as being physically transparent and analytically self-consistent for quantum confinement in ring-shaped molecular and nanostructured systems in magnetic fields.

Recommendations

The thermodynamic properties of the potential model should be investigated in future, alongside its relativistic corrections, using the Klein–Gordon or Dirac equations for diatomic and polyatomic molecules.

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